Exercise 70

Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx} f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.

(a) Find $g'(0)$ and $g''(0)$ in terms of c.

(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

Solution

Take the derivative of $g(x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}[g(x)] \\ &= \frac{d}{dx}[e^{cx} + f(x)] \\ &= \frac{d}{dx}(e^{cx}) + \frac{d}{dx}[f(x)] \\ &= e^{cx} \cdot \frac{d}{dx}(cx) + f'(x) \\ &= e^{cx} \cdot (c) + f'(x) \\ &= ce^{cx} + f'(x) \end{aligned}$$

Evaluate it at $x = 0$.

$$g'(0) = ce^0 + f'(0) = c + 5$$

Find the second derivative.

$$\begin{aligned} g''(x) &= \frac{d}{dx}[g'(x)] \\ &= \frac{d}{dx}[ce^{cx} + f'(x)] \\ &= c\frac{d}{dx}(e^{cx}) + \frac{d}{dx}[f'(x)] \\ &= c(ce^{cx}) + f''(x) \\ &= c^2 e^{cx} + f''(x) \end{aligned}$$

Evaluate it at $x = 0$.

$$g''(0) = c^2 e^0 + f''(0) = c^2 - 2$$

Plug in $x = 0$ to $h(x)$ to find the corresponding y-value for the point on the curve.

$$h(0) = e^0 f(0) = 3$$

All that's needed now is the slope at $x = 0$. Differentiate $h(x)$

$$h'(x) = \frac{d}{dx}[h(x)] = \frac{d}{dx}[e^{kx} f(x)] = \left[\frac{d}{dx}(e^{kx})\right] f(x) + e^{kx} f'(x) = (ke^{kx}) f(x) + e^{kx} f'(x)$$

and evaluate it at $x = 0$.

$$h'(0) = kf(0) + f'(0) = 3k + 5$$

Therefore, the equation of the tangent line to $h(x)$ at $(0, 3)$ is

$$y - 3 = (3k + 5)(x - 0).$$